Chapter 5.2: Sigma Notation and Limits of Finite Sums

Sigma Notation

Compressing a big sum into a compact form.

$$\sum_{k=p}^{q} a_k = a_p + a_{p+1} + a_{p+2} + \cdots + a_q$$

Index stops at $k = q$

Sigma $\sum_{k=p}^{q} a_k$ Terms in the sum depending on k

Dummy index variable k Index starts at $k = p$

Example:

$$\sum_{k=1}^{4} k = 1 + 2 + 3 + 4 \qquad \sum_{k=1}^{3} 7 = 7 + 7 + 7$$

Using $\sum$

Express the following sums in sigma notation:

- $1+2+3+4+5=\sum_{k=1}^{5} k$
- $1+3+5+7=\sum_{k=1}^{4} 2k-1$
- $-1+2-3+4-5+6=\sum_{k=1}^{6}(-1)^k k$
- $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\sum_{k=1}^{4}\frac{1}{2^k}$

Useful Formulas

- $\sum_{k=p}^{q}(a_k + b_k) = \sum_{k=p}^{q} a_k + \sum_{k=p}^{q} b_k$
- $\sum_{k=p}^{q}(c \cdot a_k) = c \cdot \sum_{k=p}^{q} a_k$
- $\sum_{k=p+r}^{q+r} (a_{k-r}) = \sum_{k=p}^{q} a_k$
- $\sum_{k=1}^{n} 1 = n$
- $\sum_{k=1}^{n} k = \frac{n \cdot (n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\sum_{k=1}^{n}(2k + 4k^3) = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 4k^3 = 2\sum_{k=1}^{n} k + 4\sum_{k=1}^{n} k^3$$
$$= 2 \cdot \frac{n \cdot (n+1)}{2} + 4 \cdot \left(\frac{n(n+1)}{2}\right)^2 = n \cdot (n+1) + (n(n+1))^2$$

Riemann Sums

Recall approximation of the area under $f(x)$ for $x \in [a, b]$
Pick $a = a_0 < a_1 < \cdots < a_n = b$

$$\text{area} \approx f(x_1)\Delta_1 + f(x_2)\Delta_2 + \cdots + f(x_n)\Delta_n = \sum_{k=1}^{n} f(x_k)\Delta_k$$

Idea: Approximation gets better as $n \to \infty$.

Take n parts of equal size and always take the right point.

$$\text{area} \approx \sum_{k=1}^{n} f(a_k)\Delta_k = \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

$$\text{Area} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \right)$$

Computing Area - simple

Find the area under $f(x) = 1$ from $x = 0$ to $x = b$ using both geometry and Riemann sums.

Geometry: Easy. It is a rectangle with width b and height 1, so it is $b \cdot 1 = b$.

Riemann sums: notice $a = 0$

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}\right) &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} 1\cdot\frac{b-0}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{b}{n}\cdot\sum_{k=1}^{n} 1\right) \\
&= \lim_{n\to\infty}\left(\frac{b}{n}\cdot n\right) \\
&= \lim_{n\to\infty} b = b
\end{aligned}$$

Computing Area - easy

Find the area under $f(x) = x$ from $x = 0$ to $x = b$ using both geometry and Riemann sums.

Geometry: Easy. It is a triangle with width b and height f(b)=b, so it is $b \cdot b/2 = b^2/2$.

Riemann sums: notice $a = 0$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(k\cdot\frac{b}{n}\right)\cdot\frac{b}{n}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} k\cdot\frac{b}{n}\cdot\frac{b}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{b^2}{n^2}\cdot\sum_{k=1}^{n} k\right) = \lim_{n\to\infty}\left(\frac{b^2}{n^2}\cdot\frac{(n+1)n}{2}\right)$$

$$= \frac{b^2}{2}\lim_{n\to\infty}\frac{n^2+n}{n^2} = \frac{b^2}{2}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1} = \frac{b^2}{2}$$

Computing Area - still the same...

Find the area under $f(x) = x^2$ from $x = 0$ to $x = b$ using Riemann sums.

Riemann sums: notice $a = 0$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(k\cdot\frac{b}{n}\right)\cdot\frac{b}{n}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^2\cdot\frac{b^2}{n^2}\cdot\frac{b}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{b^2}{n^2}\cdot\sum_{k=1}^{n} k^2\right) = \lim_{n\to\infty}\left(\frac{b^3}{n^3}\cdot\frac{n(n+1)\cdot(2n+1)}{6}\right)$$

$$= \frac{b^3}{6}\lim_{n\to\infty}\frac{2n^3+3n^2+n}{n^3} = \frac{b^3}{6}\lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1} = \frac{b^3}{3}$$

Guess for $f(x) = x^k$?

We computed:

The area under $f(x) = 1$ from $x = 0$ to $x = b$ is b.

The area under $f(x) = x$ from $x = 0$ to $x = b$ is $\frac{b^2}{2}$.

The area under $f(x) = x^2$ from $x = 0$ to $x = b$ is $\frac{b^3}{3}$.

The area under $f(x) = x^k$ from $x = 0$ to $x = b$ is $\frac{b^{k+1}}{k+1}$.

This should look familiar, like an antiderivative of x^k.

Computing Area - finally one exciting!

Find the area under $f(x) = 1 - x^2$ from $x = 0$ to $x = 1$ using Riemann sums.

$$\sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} = \sum_{k=1}^{n} f\left(0 + k \cdot \frac{1-0}{n}\right) \cdot \frac{1-0}{n}$$

$$= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left(1 - \frac{k^2}{n^2}\right)\frac{1}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) = \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= 1 - \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

Now, if we take the limit as $n \to \infty$, then we arrive at

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \frac{2n^3 + 3n^2 + n}{6n^3} = 1 - \frac{2}{6} = \frac{2}{3}$$

So, we can say certainly that the area under $f(x) = 1 - x^2$ on $[0, 1]$ is $2/3$